

# Characterizing perfect recall using next-step temporal operators in S5 and sub-S5 Epistemic Temporal Logic

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## Abstract

We review the notion of perfect recall in the literature on interpreted systems, game theory, and epistemic logic. In the context of Epistemic Temporal Logic (ETL), we give a (to our knowledge) novel frame condition for perfect recall, which is local and can straightforwardly be translated to a defining formula in a language that only has next-step temporal operators. This frame condition also gives rise to a complete axiomatization for S5 ETL frames with perfect recall. We then consider how to extend and consolidate the notion of perfect recall in sub-S5 settings, where the various notions discussed are no longer equivalent.

## 1 Introduction

Perfect recall is an epistemic-temporal notion concerning an agent’s ability to remember the past. It does not entail that all knowledge an agent currently has is preserved—in fact, certain (negative) knowledge must be lost as the agent learns. For example “I know that I don’t know  $p$ ” is lost when I learn  $p$ . Rather, it means that an agent remembers all the information he once *had*, and can use it to reason about the present.

The notion of perfect recall is well-studied in game theory (see, e.g., [14] and [10, Section 11.1.3]) and in the distributed computing/interpreted systems/temporal logic literature [4, 19, 20, 8, 13] (see [6] for a discussion from the viewpoint of the intersection of both). In recent years, it has also been discussed in the epistemic logic community [16, 17, 18, 9], specifically in the context of Epistemic Temporal Logic (ETL) (see, e.g., [17]). ETL is an epistemic logic (or rather, family of logics) with added event modalities, interpreted on tree models, and it is intended to capture interactions of agents over time. Unlike with interpreted systems, the motivation is not dominated by the idea of processes with control programs which govern their behavior into infinity, and therefore, ETL does not necessarily aim at making statements about such long-term behavior. Although the concept of histories (corresponding to runs in interpreted systems)

does exist and variants of ETL with long-range operators such as “until” have been studied, common ETL languages focus on the local perspective and lack global temporal operators that can talk about indefinite time spans. In the following, we use “ETL” in the narrow sense of epistemic temporal logic with only next-step (local) temporal modalities.

Due to the differences between the mentioned frameworks, various definitions and conditions for perfect recall exist. Our aim here is to review and examine the relevant ones from the viewpoint of a logic without long-range temporal operators. We use the history-based framework and notation of ETL (also used by Parikh and Ramanujam [13]), but interpreted systems are for our purposes essentially equivalent [11] and so our considerations should be transferable, *mutatis mutandis*.

In synchronous settings (i.e., where agents have access to a global clock), there is a well-known characterization of perfect recall, including a simple frame condition and a definition that can be expressed in ETL. In contrast, without synchronicity the existing characterizations of perfect recall seem to be inherently global in the sense that they talk about whole (prefixes of) histories (or runs), or refer to some arbitrary point in them. This precludes a direct translation into a first-order frame condition (since reachability is not definable in first-order logic) and also makes it unclear whether the notion can be expressed in ETL. This is not an issue for the interpreted systems community, since they by default have long-range temporal operators available.<sup>1</sup> Indeed, van der Meyden [19] axiomatized perfect recall using the “until” operator. However, this does not transfer to logics which only have next-step temporal operators.

After briefly introducing ETL in Section 2 and discussing perfect recall from the perspective of ETL in Section 3, we therefore in Section 4 propose an alternative characterization of perfect recall which only uses single steps of temporal (and epistemic) succession. We show that it is first-order definable and definable in ETL, and we give a complete axiomatization for S5 ETL with perfect recall. We then explore sub-S5 settings in Section 5. Section 6 concludes.

## 2 Epistemic-temporal logic (ETL)

We focus on the single-agent case since perfect recall is a property inherent to one agent; all our considerations carry over to the multi-agent case. We consider models over some finite set  $E$  of **events**. A **history**  $h \in E^*$  is a finite sequence of events, and we denote the empty history by  $\epsilon$ . We denote **sequences** simply by listing their elements, possibly preceded by a prefix sequence. For two histories  $h, h'$  we write  $h \rightsquigarrow_e h'$  if  $h' = he$ , that is, if  $h'$  **extends**  $h$  by one event  $e$ . We write  $h \rightsquigarrow h'$  if  $h \rightsquigarrow_e h'$  for some event  $e$ . We denote the transitive and reflexive closure of  $\rightsquigarrow$  by  $\rightsquigarrow^*$ , so  $h \rightsquigarrow^* h'$  says that  $h$  is a **prefix** of  $h'$  (possibly  $h'$  itself), or vice versa,  $h'$  is an **extension** of  $h$ . For  $h \rightsquigarrow^* h'$ , we sometimes also write

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<sup>1</sup>It is not an issue for the game theory community because they do not commonly concern themselves with formal languages.

$h \preceq h'$ , and  $h \prec h'$  for  $h \preceq h'$  with  $h \neq h'$ . A **protocol**  $H \subseteq E^*$  is a finite set of histories closed under taking prefixes, intuitively representing the allowed evolutions of the system.

An **(epistemic) accessibility relation** is a binary relation  $\sim \subseteq H \times H$  on histories. It specifies, for any given history  $h$ , the histories that the agent **considers possible** at  $h$ . Various conditions can be imposed on  $\sim$ , making it capture various notions of knowledge or belief (for details see, e.g., [2]). Most commonly, the relation is assumed to be an *equivalence relation*, making it capture a notion of “correct knowledge”. This case is also referred to as **S5**. We refer to less restrictive cases as **sub-S5**. Relevant properties include *transitivity*, reflecting **positive introspection**, and *Euclideaness*, reflecting **negative introspection**. Another property is that of **synchronicity**, which intuitively means that the agent has access to a clock. It holds if, whenever  $h \sim h'$ , then the lengths of  $h$  and  $h'$  are equal.

An **ETL frame** is a tuple  $\mathcal{F} = \langle E, H, \sim \rangle$  consisting of a set of events  $E$ , a protocol  $H$  and an epistemic accessibility relation  $\sim$ . We will usually omit  $E$  and  $H$  for the sake of clarity and implicitly assume that any events or histories we talk about belong to  $E$  or  $H$ , respectively. An ETL frame can be viewed as a temporal **tree** (induced by the  $\rightsquigarrow$  relation) with epistemic accessibilities between nodes. We will also consider an extension to **forests**, which are (finite) sets of trees with distinct roots and possibly interrelating epistemic accessibilities.<sup>2</sup> Most of our considerations apply both to trees and to forests, and only in Section 5 we have to make the distinction explicit. We use properties of the epistemic accessibility relation to specify frames with a corresponding relation; for example, by an *S5 frame* we mean a frame with an S5 accessibility relation.

The **language** of ETL consists of a finite set **At** of propositional atoms and of all formulas built from those according to the following grammar:

$$p \mid \neg\varphi \mid \varphi \wedge \psi \mid K\varphi \mid \langle e \rangle\varphi,$$

where  $p \in \mathbf{At}$  and  $\varphi, \psi$  are formulas. Intuitively,  $K\varphi$  means that the agent knows  $\varphi$ , and  $\langle e \rangle\varphi$  means that event  $e$  can occur and afterwards  $\varphi$  will hold. The remaining propositional connectives are defined as abbreviations as usual, and the duals of the modalities are denoted by  $L$  (dual of  $K$ ) and  $[e]$  (dual of  $\langle e \rangle$ ). We write  $\Diamond\varphi$  to abbreviate  $\bigvee_{e \in E} \langle e \rangle\varphi$ .

A **valuation**  $V : \mathbf{At} \rightarrow 2^H$  assigns to each atom the set of histories where it is true. We write  $\mathcal{F}, V, h \models \varphi$  for a frame  $\mathcal{F}$ , a valuation  $V$  and a history  $h$  of the protocol of  $\mathcal{F}$  to say that  $\varphi$  is **satisfied** by  $\mathcal{F}, V, h$ . Satisfaction of formulas is defined inductively as usual, starting with the truth values of atoms as given

<sup>2</sup>The distinction between ETL frames and forests corresponds to the *unique initial state* condition in the interpreted systems literature.

by  $V$ , and with the following semantics:

$\mathcal{F}, V, h \models p$	iff $h \in V(p)$
$\mathcal{F}, V, h \models \neg\varphi$	iff $\mathcal{F}, V, h \not\models \varphi$
$\mathcal{F}, V, h \models \varphi \wedge \psi$	iff $\mathcal{F}, V, h \models \varphi$ and $\mathcal{F}, V, h \models \psi$
$\mathcal{F}, V, h \models K\varphi$	iff for each $h' \in H$ with $h \sim h'$ : $\mathcal{F}, V, h' \models \varphi$
$\mathcal{F}, V, h \models \langle e \rangle \varphi$	iff $he \in H$ and $\mathcal{F}, V, he \models \varphi$

A formula  $\varphi$  is said to be **valid** on  $\mathcal{F}, V$  if and only if  $\mathcal{F}, V, h \models \varphi$  for all  $h \in H$ . It is said to be valid on  $\mathcal{F}$  if and only if it is valid on  $\mathcal{F}, V$  for all valuations  $V$ . It is said to **define** a class  $C$  of frames if and only if it is valid exactly on the frames in  $C$ .

For a binary relation  $R$  on histories we write  $[h]_R = \{h' \mid hRh'\}$  to denote the **image** of  $h$  under  $R$ . If  $H$  is a set of histories, we write  $[H]_R$  for  $\bigcup_{h \in H} [h]_R$ . Obviously, if  $R$  is an equivalence relation, then  $[h]_R$  is the equivalence class of  $h$  with respect to  $R$ .

### 3 Perfect recall

In this section, we review existing definitions of perfect recall, give intuitive justifications for the notions, and examine how they relate to each other.

Like in distributed systems, and unlike in game theory, in ETL there is no notion of turns and no notion of agency associated with events. An event is just an event and comes with no specification as to who performs it. In game theory, turn-taking typically makes successive situations distinguishable and perfect recall can often be formulated as “remembers all his actions”. In ETL, these features are not part of the model<sup>3</sup> and so we have to use other ways to express the notion.

Whether or not time is part of the agent’s perception is exactly what makes the difference between asynchronous and synchronous systems. Synchronicity can be defined separately, if it is desired; we are interested in perfect recall as an independent property, which does not interfere with (a)synchronicity, and thus should not assume or imply that agents perceive time.

#### 3.1 Basic definitions

We start by giving some intuition about the notions we are going to define. As mentioned, an agent with perfect recall can at any point remember all the information that he had at any previous point in time, and is able to exclude any possibilities for the current state of the world which are inconsistent with that information.

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<sup>3</sup>Put differently, events may perfectly well be caused externally and go completely unnoticed by any agent. If we do want to attribute certain events to certain agents (a view which certainly suggests itself in the single-agent case, and is also customary in game theory), then corresponding observability conditions for the agent performing a particular action can be specified separately, and our definitions of perfect recall will not interfere.

**Two different intuitions for perfect recall** There are two related ways in which an agent might detect such inconsistencies, the first on the level of epistemic states, and the second on the level of the semantic structures that model them.<sup>4</sup> Consider a perfect-recall agent in some state of the world, in ETL terms a history,  $h$ , and some other history  $h'$ .

Firstly, if in state  $h'$  the agent would have gone through a different sequence of epistemic states than he actually has in  $h$ , then he can exclude the possibility of  $h'$ , since he can recall all his epistemic states.

Secondly, if in the state before  $h$  the agent was certain that the world was *not* in a state along history  $h'$ , then at  $h$  the agent can exclude the possibility that the world is in state  $h'$ , since he can recall his previous assessments. Put differently, if  $h'$  is *not* an extension of some history considered possible before, then the agent can exclude the possibility of  $h'$  since there would have been no way for the world to evolve to  $h$ .

We formalize these two intuitions in the following. As we will see, they are equivalent in the context of S5, while in the general case, neither implies the other (see Section 5).

**Formalizing the intuitions** The first notion is the one most commonly used as starting point in the literature. It uses the idea of local-state sequences, or “epistemic experiences”, meaning sequences of epistemic states that the agent has gone through. Repetitions of identical states are ignored, since the agent has no way of discriminating between two states in which he has the same epistemic state. As in game theory (cf. [10, Section 11.1.3]), we identify an agent’s epistemic state with his information set, i.e., the set of accessible worlds.

**Definition 3.1.** Given an ETL frame and a history  $e_1 \dots e_\ell$ , the agent’s **epistemic experience** is the sequence

$$\text{EE}(e_1 \dots e_\ell) := [\epsilon] \sim [e_1] \sim [e_1 e_2] \sim \dots [e_1 \dots e_\ell] \sim$$

of epistemic states (or information sets, in game theory parlance) he has gone through.

We say that the epistemic experiences in two histories  $h, h'$  are **equivalent modulo stutterings**, in symbols  $\text{EE}(h) \approx \text{EE}(h')$ , if and only if the sequences with all repetitions of subsequent identical sets removed are equivalent.

An ETL frame has **perfect recall with respect to epistemic experience** ( $\text{PR}_{\text{ee}}$ ) if and only if, whenever  $h \sim h'$ , we have  $\text{EE}(h) \approx \text{EE}(h')$ .

The second definition is a slight (but equivalent) variant of a notion most commonly used in the interpreted systems literature.<sup>5</sup> In the literature it has been mostly used as a technical condition characterizing  $\text{PR}_{\text{ee}}$ , but by rephrasing it we can provide it with an independent motivation.

<sup>4</sup>See Section 6 for some discussion related to the question of which aspects of the model an agent can access.

<sup>5</sup>See [7, p. 204] (who call perfect recall “no forgetting”) and [19, Proposition 2.1(a)].

**Definition 3.2.** An ETL frame has **perfect recall with respect to history consistency** ( $\text{PR}_{\text{hc}}$ ) if and only if for any histories  $h, h'$  and event  $e$  with  $he \sim h'$ , there is some history  $h''$  with  $h \sim h'' \rightsquigarrow^* h'$  (i.e., some prefix of  $h'$  is epistemically accessible from  $h$ ).

Put differently, the condition is that for each history  $h$  and event  $e$ , we have

$$[he]_{\sim} \subseteq [[h]_{\sim}]_{\rightsquigarrow^*} .$$

This second formulation suggests an intuitive reading: A frame has  $\text{PR}_{\text{hc}}$  if all histories considered possible after some event are extensions of histories considered possible before the event.

We refine this notion in Section 4 to obtain one that is more fine-grained and detects more inconsistencies in sub-S5 settings; however, in the context of S5 the notions are equivalent, so for simplicity we stick with this definition for now.

There are two further related conditions in the literature, which we will state next.

**Definition 3.3** (cf. [20, 18]). An ETL frame has **synchronous perfect recall** ( $\text{sPR}$ ) if and only if for each  $h, h', e$  with  $he \sim h'$  there is  $h''$  with  $h \sim h'' \rightsquigarrow h'$ . Put differently, the condition is that for each history  $h$  and event  $e$ , we have

$$[he]_{\sim} \subseteq [[h]_{\sim}]_{\rightsquigarrow} .$$

That is, all histories considered possible after some event are extensions of histories considered possible before the event *by exactly one event*.

**Definition 3.4** (cf. [18, p. 503, Definition 11]). An ETL frame has **weak synchronous perfect recall** ( $\text{wsPR}$ ) if and only if for all  $h, h', e, e'$  with  $he \sim h'e'$ , we have  $h \sim h'$ . Put differently, for each history  $h$  and event  $e$  we have

$$[he]_{\sim} \subseteq [[h]_{\sim}]_{\rightsquigarrow} \cup \{\epsilon\} .$$

It seems difficult to find an intuitive justification for this last definition: Why should an agent with perfect recall be characterized to consider possible, after some event, one-step extensions of histories previously considered possible *or the empty history*? In order to get a better understanding of the various notions, we now take a closer look at how they relate.

### 3.2 Relating the notions

First of all, as mentioned above, the two notions of  $\text{PR}_{\text{ee}}$  and  $\text{PR}_{\text{hc}}$  are equivalent in S5.

**Proposition 3.5.** *An S5 ETL frame has  $\text{PR}_{\text{ee}}$  if and only if it has  $\text{PR}_{\text{hc}}$ .*

*Proof.* This follows immediately from [19, Proposition 2.1], instantiating what the interpreted systems literature calls “local states” by the set of accessible worlds in ETL frames.

To give a version of the proof, we consider two histories and show that the conditions of  $\text{PR}_{\text{ee}}$  and  $\text{PR}_{\text{hc}}$  are equivalent. We first show that  $\text{PR}_{\text{ee}}$  implies  $\text{PR}_{\text{hc}}$ . For two empty histories this is obvious, so w.l.o.g. we assume that the first history is non-empty. So assume that  $\text{PR}_{\text{ee}}$  holds and that  $he \sim h'$ . We have either of two cases:

- (i)  $[h]_{\sim} = [he]_{\sim}$ . But then  $h \sim h'$ , and  $\text{PR}_{\text{hc}}$  is satisfied.
- (ii)  $[h]_{\sim} \neq [he]_{\sim}$ . By  $\text{PR}_{\text{ee}}$ , we must have  $\text{EE}(he) \approx \text{EE}(h')$ , and thus there must be  $h'' \rightsquigarrow h'$  with  $[h]_{\sim} = [h'']_{\sim}$ . So again,  $\text{PR}_{\text{hc}}$  is satisfied.

To see that  $\text{PR}_{\text{hc}}$  implies  $\text{PR}_{\text{ee}}$ , we proceed by induction on the sum of the lengths of the two histories. The base case with both empty is straightforward. For the induction step, assume that  $\text{PR}_{\text{hc}}$  holds. W.l.o.g. we assume that the first history is non-empty, so we consider  $he$  and  $h'$  for some event  $e$ , with  $he \sim h'$ .  $\text{PR}_{\text{hc}}$  yields one of two cases:

- (i)  $h \sim h'$ . Then the induction hypothesis yields  $\text{EE}(h) \approx \text{EE}(h')$ . Furthermore, we have  $[h]_{\sim} = [h']_{\sim} = [he]_{\sim}$ , so  $\text{EE}(h) \approx \text{EE}(he)$ . Taken together, we obtain  $\text{EE}(he) \approx \text{EE}(h')$ , and  $\text{PR}_{\text{ee}}$  is satisfied.
- (ii)  $h \sim h''$  for some  $h'' \prec h'$ . It then follows that  $h'$  is non-empty. Let  $g$  be its direct predecessor, i.e.,  $g \rightsquigarrow h'$ . From  $h' \sim he$ , with  $\text{PR}_{\text{hc}}$  it follows that there is  $h''' \preceq he$  such that  $g \sim h'''$ . If  $h''' = he$ , then  $\text{PR}_{\text{ee}}$  follows analogously as in the previous case. Otherwise we have  $h''' \preceq h$ . The induction hypothesis yields  $\text{EE}(h) \approx \text{EE}(h'')$  as well as  $\text{EE}(g) \approx \text{EE}(h''')$ . Since  $h'' \preceq g$  and  $h''' \preceq h$ , we must have  $\text{EE}(h) \approx \text{EE}(g)$ . Since  $h \rightsquigarrow he$  and  $g \rightsquigarrow h'$ , together with  $[he]_{\sim} = [h']_{\sim}$  we obtain that  $\text{EE}(he) \approx \text{EE}(h')$ .  $\square$

As we see in the following, the remaining two notions,  $\text{sPR}$  and  $\text{wsPR}$ , are similar to these but impact another property, namely that of *synchronicity*.  $\text{PR}_{\text{ee}}$  and  $\text{PR}_{\text{hc}}$ , on the other hand, do not interfere with synchronicity. In agreement with the interpreted systems and game theory literature, we therefore use them as fundamental definitions of perfect recall in the context of S5, and due to their equivalence, we use **perfect recall (PR)** to refer to both.

In the presence of synchronicity, not surprisingly, all notions are equivalent.

**Proposition 3.6.** *On synchronous S5 ETL frames, all of the above notions ( $\text{PR}_{\text{ee}}$ ,  $\text{PR}_{\text{hc}}$ ,  $\text{sPR}$ ,  $\text{wsPR}$ ) are equivalent.*

*Proof.* Straightforward with Proposition 3.5; see also [8, 20].  $\square$

It is easy to see that  $\text{sPR}$  implies synchronicity, since  $\rightsquigarrow$  is well-founded.<sup>6</sup> Therefore, in general on S5 frames,  $\text{sPR}$  is equivalent to  $\text{PR}$  *plus* synchronicity. As for  $\text{wsPR}$ , the relationship is a bit less clear-cut.

<sup>6</sup>One formulation of  $\text{sPR}$  is (cf. Proposition 3.9): “If the agent knows that at the next time point  $p$  holds, then at the next time point he knows that  $p$  holds.” Intuitively this implies that the mere ticking of the clock affects the agent’s mental state.

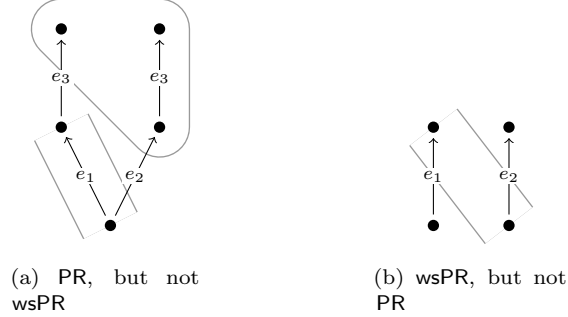


Figure 1: Two frames, gray lines indicating information sets. (a) **wsPR** is incomplete with respect to **PR** (**wsPR** is violated because  $e_1e_3 \sim e_2e_3$  but  $e_1 \not\sim e_2$ ), and (b) **wsPR** is not sound with respect to **PR** on ETL forests: Intuitively, event  $e_1$  lets the agent “forget” that he is in the left tree. Note that on forests the definition of **wsPR** has to be adjusted by replacing  $\{\epsilon\}$  by the set of all roots.

**Proposition 3.7.** *On S5 ETL frames, **wsPR** implies **PR**, but not vice versa. Put differently, **wsPR** is sound but incomplete with respect to **PR**.*

*Proof.* Soundness is straightforward; for incompleteness, see Figure 1(a).  $\square$

In that sense, **wsPR** is “somewhere in between” **sPR** and **PR**: It is strictly implied<sup>7</sup> by **sPR** and strictly implies **PR**. What distinguishes **wsPR** is that, while it (unlike **sPR**) does not *imply* synchronicity, it does (unlike **PR**) *presuppose* it, in that it fails to classify asynchronous frames correctly—given that, as we have argued, **PR** captures the intuition of perfect recall as an independent property. We therefore neglect **wsPR** in the further discussion: Synchronous perfect recall frames are naturally captured by **sPR** and asynchronous ones by **PR**, while **wsPR** has no clear domain of application in our context.

*Remark 3.8.* Except where noted, all our considerations carry over to ETL forests (sets of ETL frames with possibly interrelating indistinguishabilities). The first such note is the fact that on forests, **wsPR** is not only incomplete with respect to **PR** but also not sound, as shown in Figure 1(b).

Note that the “local histories” of Parikh and Ramanujam [13] give rise to frames in which exactly those histories are indistinguishable in which the epistemic experiences are equivalent modulo stutterings, so that their frames inherently satisfy **PR**.<sup>8</sup> The version used by Pacuit et al. [12] analogously produces frames which inherently satisfy **sPR** (and thus synchronicity).

Finally, we note that **sPR** is definable in ETL in a neat way, which also gives rise to an axiomatization.

<sup>7</sup>By this we mean “implies, but is not equivalent to”.

<sup>8</sup>In fact, the frames they obtain are exactly those S5 frames which satisfy **PR** and a “fixed observability” condition stating that, for each agent, there is a fixed set of events whose occurrence he is able to distinguish from the (non-)occurrence of others.



**Proposition 3.9.** *An S5 ETL frame has sPR if and only if it validates the following formula:*

$$\Diamond Lp \rightarrow L\Diamond p$$

*Together with S5, this axiomatizes ETL on synchronous S5 frames with perfect recall.*

This result is well-known and is analogous to [20, Theorem 4] and [8, Theorem 3.6], since sPR characterizes synchronous frames with perfect recall.<sup>9</sup> It can be proved using Sahlqvist [15], similar to the proof of Theorem 4.4.

A corresponding characterization of PR has been given by van der Meyden [19], but that characterization uses the “until” operator. To our knowledge, a version using only “short-sighted” next-step temporal modalities has not been discussed. Providing it is the aim of the next section.

## 4 Defining perfect recall in ETL

We start by giving a “local” version of perfect recall with respect to history consistency, whose intuitive interpretation gives a more fine-grained account of the agent’s possibilities for reasoning than  $\text{PR}_{\text{hc}}$  does. As we will see in Section 5, in general the notions are not equivalent, and the fine-grainedness of the version we propose indeed makes a difference. In the current section, however, we show that in the context of S5 this definition is equivalent to the preceding ones, and we exploit its locality to find an axiomatization of perfect recall using only next-step temporal modalities.

**Definition 4.1.** An ETL frame has  $\text{PR}_{\text{hc}}$ , **local version** ( $\text{PR}_{\text{hc}}^\ell$ ) if and only if for each history  $h$  and event  $e$ , we have

$$[he]_{\sim} \subseteq [h]_{\sim} \cup [[h]_{\sim}]_{\rightsquigarrow} \cup [[he]_{\sim}]_{\rightsquigarrow}$$

Put differently, for each  $h, h', e$  with  $he \sim h'$ , either of the following holds:

- (i)  $h \sim h'$
- (ii)  $h \sim h'' \rightsquigarrow h'$  for some  $h''$
- (iii)  $he \sim h'' \rightsquigarrow h'$  for some  $h''$ .

See Figure 2 for an illustration of this definition. Let us walk through it and compare it with  $\text{PR}_{\text{hc}}$ . We consider the history  $he$  and refer to  $h$  as “before  $e$ ” and to  $he$  as “after  $e$ ”. The definition says that any history considered possible after  $e$  either

- (i) was considered possible already before  $e$ , i.e., the agent didn’t notice  $e$ , nor time passing; or

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<sup>9</sup>Compare also the Cross Axiom of Dabrowski et al. [3].

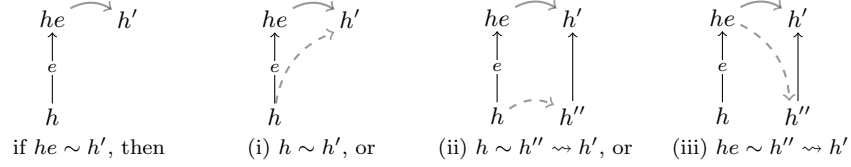


Figure 2: Illustrating  $\text{PR}_{\text{hc}}^\ell$ . Gray, bent arrows indicate accessibilities, showing the directed versions for the sake of illustration.

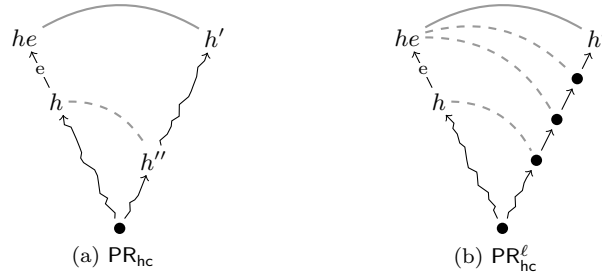


Figure 3:  $\text{PR}_{\text{hc}}^\ell$  inductively bridges the gap to  $\text{PR}_{\text{hc}}$ , but it imposes additional conditions on the intermediate states. However, in S5 the two conditions are equivalent.

(ii) is an extension by one event of a history considered possible before  $e$ , i.e., the agent correctly thinks one event occurred, though he may not be certain which one; or

(iii) is an extension by one event of another history considered possible after  $e$ .

As illustrated in Figure 3, this last condition inductively bridges the gap to  $\text{PR}_{\text{hc}}$ , and allows the agent to consider possible that several events occur while really just  $e$  is happening. The difference, as compared to  $\text{PR}_{\text{hc}}$ , is that now there is a stricter consistency requirement. If the agent considers possible that several events have happened, he must obviously be unable to detect some of them, since really just one event happened. Given that, he must also consider possible the intermediate histories along these several events—either from the history after  $e$  or from before  $e$ . Exactly this consistency requirement is inductively captured by the last condition (in interplay with the second condition).

Intuitively it is thus clear that  $\text{PR}_{\text{hc}}^\ell$  is at least as strong a condition as  $\text{PR}_{\text{hc}}$ , as formalized in the next result.

**Lemma 4.2.** *Any ETL frame that has  $\text{PR}_{\text{hc}}^\ell$  also has  $\text{PR}_{\text{hc}}$ .*

*Proof.* A simple induction on the length of  $h'$  in the definition of  $\text{PR}_{\text{hc}}^\ell$  proves the claim.  $\square$

In Section 5 we will see that  $\text{PR}_{\text{hc}}^\ell$  is in general strictly stronger than  $\text{PR}_{\text{hc}}$ ; and in particular in Figure 4 that it detects situations where (intuitively) information is lost, for which  $\text{PR}_{\text{hc}}$  fails to do so. However, in the context of S5, the notions are in fact equivalent.

**Proposition 4.3.** *An S5 ETL frame has PR if and only if it has  $\text{PR}_{\text{hc}}^\ell$ .*

*Proof.* Recall that we use PR to refer to  $\text{PR}_{\text{hc}}$  and/or  $\text{PR}_{\text{ee}}$ , since they are equivalent according to Proposition 3.5.  $\text{PR}_{\text{hc}}^\ell$  implies  $\text{PR}_{\text{hc}}$  by Lemma 4.2. Vice versa, a simple induction on the length of  $h'$  shows that  $\text{PR}_{\text{hc}}$  and  $\text{PR}_{\text{ee}}$  imply  $\text{PR}_{\text{hc}}^\ell$ .  $\square$

Thus, in the context of S5,  $\text{PR}_{\text{hc}}^\ell$  is equivalent to the “established” notions and we can use it to characterize perfect recall frames.

**Theorem 4.4.** *An ETL frame has  $\text{PR}_{\text{hc}}^\ell$  if and only if it validates the following formula for each event  $e$  (recall that  $\Diamond\varphi$  abbreviates  $\bigvee_{e'}\langle e'\rangle\varphi$ ):*

$$\langle e\rangle Lp \rightarrow Lp \vee L\Diamond p \vee \langle e\rangle L\Diamond p . \quad (\star)$$

*Together with the normal modal logic axioms and deduction rules, it is sound and complete with respect to the class of ETL frames with  $\text{PR}_{\text{hc}}^\ell$ .*

*Proof.* This follows from the Correspondence and Completeness theorems by Sahlqvist [15]. In some more detail, note first that the formula as frame property is equivalent to

$$Kp \wedge K\Box p \wedge [e]K\Box p \rightarrow [e]Kp ,$$

where  $\Box\varphi$  abbreviates  $\bigwedge_{e'}[e']\varphi$ . This is a Sahlqvist formula, so as an axiom it is complete with respect to the class of frames it defines. To see what that class is, we start from the second-order formulation of this frame property (for any histories  $h, h'$  and event  $e'$  we write  $h \rightsquigarrow_{e'} h'$  if and only if  $he' = h'$ , and we write  $h \rightsquigarrow h'$  if and only if  $h \rightsquigarrow_{e'} h'$  for some  $e'$ ):

$$\begin{aligned} & \forall P \forall h_1 [ \forall h_2 (h_1 \sim h_2 \rightarrow Ph_2) \\ & \quad \wedge \forall h_2 \forall h_3 (h_1 \sim h_2 \rightsquigarrow h_3 \rightarrow Ph_3) \\ & \quad \wedge \forall h_2 \forall h_3 \forall h_4 (h_1 \rightsquigarrow_e h_2 \sim h_3 \rightsquigarrow h_4 \rightarrow Ph_4) ] \\ & \rightarrow \forall h_2 \forall h_3 (h_1 \rightsquigarrow_e h_2 \sim h_3 \rightarrow Ph_3) \end{aligned}$$

As Sahlqvist pointed out, since  $P$  does not occur negated in the consequent, the minimal instantiation of  $P$  satisfying the antecedent yields an equivalent first-order formula. This minimal instantiation can be read off as:

$$Ph := h_1 \sim h \vee \exists h_2 (h_1 \sim h_2 \rightsquigarrow h) \vee \exists h_2 \exists h_3 (h_1 \rightsquigarrow_e h_2 \sim h_3 \rightsquigarrow h) .$$

Since it satisfies the antecedent, we are left with the instantiated consequent:

$$\begin{aligned} & \forall h_1 \forall h_2 \forall h_3 [ h_1 \rightsquigarrow_e h_2 \sim h_3 \rightarrow \\ & \quad h_1 \sim h_3 \\ & \quad \vee \exists h_4 (h_1 \sim h_4 \rightsquigarrow h_3) \\ & \quad \vee \exists h_4 \exists h_5 (h_1 \rightsquigarrow_e h_4 \sim h_5 \rightsquigarrow h_3) ] . \end{aligned}$$

Since there is at most one  $e$ -successor for any given history  $h$  (namely  $he$ ), we can replace  $h_4$  in the last disjunct by  $h_2$ . It is then easy to see that this is equivalent to  $\text{PR}_{\text{hc}}^\ell$ .  $\square$

**Corollary 4.5.** *An S5 ETL frame has PR if and only if it validates  $(\star)$ .*

*Proof.* Immediate with Proposition 4.3 and Theorem 4.4.  $\square$

We thus have an axiomatization of S5 ETL with perfect recall. However, some of the results we used indeed depended on S5. If we give up S5, we have to take a fresh look at certain issues, and that is the topic of the next section.

## 5 Sub-S5 settings

To our knowledge, perfect recall has only been considered in the context of S5 in the literature, a likely reason being that both communities that have studied the notion most (interpreted systems and game theory) virtually exclusively consider S5 settings. However, it may make perfect sense, for example, to say of a misinformed agent that he correctly remembers all information he has ever had, even if that information itself is not correct. In this section, we explore such settings of general ETL frames.

### 5.1 Preliminaries

We stick with the symmetric-looking symbol  $\sim$  even when the relation is not necessarily symmetric. Note that  $[h]_\sim$  now does not necessarily contain  $h$  itself anymore, but we have the following fact.

**Fact 5.1.** *If  $\sim$  is a transitive and Euclidean relation, then it is an equivalence relation on  $[h]_\sim$  for any  $h$  (cf. [5, Theorem 3.3]). Consequently, for any  $h, h'$  with  $[h]_\sim \cap [h']_\sim \neq \emptyset$ , we have  $[h]_\sim = [h']_\sim$ .*

Note that the motivation and justifications for the definitions of perfect recall we gave did not assume S5 knowledge. Each of the notions captured a particular way of not losing information, and they still make sense without S5. We therefore take over the basic definitions without any change, but take a new look at how they relate.

### 5.2 Contrasting the notions

First note that  $\text{PR}_{\text{hc}}^\ell$  still implies  $\text{PR}_{\text{hc}}$ , since Lemma 4.2 did not assume S5. However, without any assumptions about the frames, none of the other mutual implications among  $\text{PR}_{\text{hc}}$ ,  $\text{PR}_{\text{hc}}^\ell$  and  $\text{PR}_{\text{ee}}$  remain. This is witnessed by Figure 4, illustrating that epistemic experience and history consistency reflect two different ways of remembering past information.

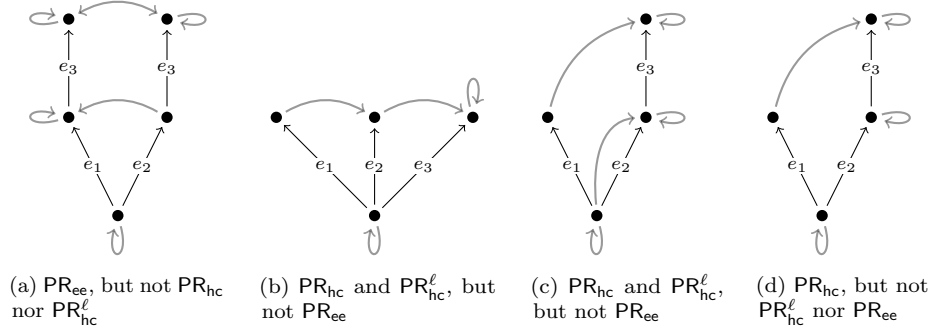


Figure 4:  $\text{PR}_{\text{hc}}$ ,  $\text{PR}_{\text{hc}}^{\ell}$ , and  $\text{PR}_{\text{ee}}$  compared on general ETL frames, gray arrows depicting the accessibilities.  $\text{PR}_{\text{hc}}$  is violated in (a) since  $e_1e_3 \sim e_2e_3$  but there is no prefix of  $e_2e_3$  that is accessible from  $e_1$ .  $\text{PR}_{\text{ee}}$  is violated in (b) since  $\text{EE}(e_1) \not\approx \text{EE}(e_2)$ , and in (c) and (d) since  $\text{EE}(e_1) \not\approx \text{EE}(e_2e_3)$ .  $\text{PR}_{\text{hc}}^{\ell}$  is violated in (d) since  $e_1 \sim e_2e_3$  but  $\epsilon \not\sim e_2e_3$  and  $\epsilon \not\sim e_2$  and  $e_1 \not\sim e_2$ .

- (a) An agent that only has perfect recall with respect to epistemic experience may at some point be certain that a particular history can be excluded, but later on “forget” this piece of information. In particular, at  $e_1e_3$  the agent considers  $e_2e_3$  possible, even though he never considered  $e_2$  possible.
- (b) and (c) An agent that only has perfect recall with respect to history consistency may at some state consider another state possible, although in that other state his epistemic experience would have been different. For example, an agent at  $e_1$  in (c) is (mistakenly) certain that he is at  $e_2e_3$ , even though in that state his previous information set would have been  $\{e_2\}$ , which contradicts his actual epistemic experience.
- (d) The intuition is similar to the previous case, but here we see that  $\text{PR}_{\text{hc}}^{\ell}$  is more fine-grained than  $\text{PR}_{\text{hc}}$ . An agent at  $e_1$  thinks he is at  $e_2e_3$ , even though he never considered  $e_2$  possible. He thus thinks himself at the endpoint of a history whose unfolding he deemed impossible.  $\text{PR}_{\text{hc}}$  grants this agent the label of perfect recall, while  $\text{PR}_{\text{hc}}^{\ell}$  denies it.

Note that, while these phenomena reflect some kind “forgetting”, they do not at first glance constitute a coherent, rational method of *belief revision*. A full-fledged doxastic logic is needed in order to really model agents that reconsider their previous assessments and deal with “unwanted” memories properly.

The following straightforward result enables us to identify the settings in which the different notions of perfect recall can be meaningfully compared.

**Proposition 5.2.** *Any ETL frame that has  $\text{PR}_{\text{ee}}$  is transitive and Euclidean.*

*Proof.* This is obvious from Definition 3.1. For example, for any three histories  $h, h', h''$ , if  $h \sim h'$  and  $h \sim h''$ , then  $\text{PR}_{\text{ee}}$  implies that  $[h]_{\sim} = [h']_{\sim} = [h'']_{\sim}$ . Since  $h', h'' \in [h]_{\sim}$ , we also get  $h' \in [h']_{\sim}$  and  $h'' \in [h']_{\sim}$ .  $\square$

This result is not very surprising, given that  $\text{PR}_{\text{ee}}$  requires of an agent to be able to assess his own epistemic experience—including at the current state. It implies that any reflexive ETL frame with  $\text{PR}_{\text{ee}}$  is already an S5 ETL frame. On the level of agents, an agent who has correct beliefs and  $\text{PR}_{\text{ee}}$  has in fact already full (S5) knowledge. However, perfect recall does not *require* the agent to have correct beliefs (in fact, perfect recall by itself is compatible with believing falsum). For example, KD45 is a common sub-S5 setting in which perfect recall is a meaningful notion.

Given that transitivity and Euclideaness are inherent to  $\text{PR}_{\text{ee}}$ , we continue our comparison within the corresponding class of frames. In the following, we use **introspective** to mean “transitive and Euclidean”.

### 5.3 Characterizing $\text{PR}_{\text{ee}}$ locally

Given the fact that  $\text{PR}_{\text{hc}}^{\ell}$  no longer characterizes  $\text{PR}_{\text{ee}}$  and the original definition of  $\text{PR}_{\text{ee}}$  is somewhat unwieldy, it can be useful to have a local condition on histories and accessibilities that corresponds to it. It turns out that we can re-use  $\text{PR}_{\text{hc}}^{\ell}$  by slightly modifying the frame in question.

For a frame  $\mathcal{F}$  with accessibility relation  $\sim$ , we will use  $\dot{\mathcal{F}}$  and  $\dot{\sim}$  to denote the S5 closure. We need the following small technical condition: We say that a frame satisfies **persistent insanity** if, whenever  $[h]_{\sim} = \emptyset$  and  $h \preceq h'$ , then  $[h']_{\sim} = \emptyset$ . Intuitively, once a corresponding agent has inconsistent beliefs, he will remain in that pitiful condition forever.

**Proposition 5.3.** *An introspective ETL frame satisfying persistent insanity has  $\text{PR}_{\text{ee}}$  if and only if its S5 closure has  $\text{PR}_{\text{hc}}^{\ell}$ .*<sup>10</sup>

*Proof.* Due to Proposition 4.3,  $\text{PR}_{\text{ee}}$  and  $\text{PR}_{\text{hc}}^{\ell}$  are equivalent on the S5 closure. We can thus prove the claim by showing that  $\text{PR}_{\text{ee}}$  is invariant under taking this closure.

To see that this is indeed the case, take any pair  $h, h'$  of histories, the accessibility relation  $\sim$  of an introspective frame satisfying persistent insanity, and its S5 closure  $\dot{\sim}$ . With Fact 5.1, it is easy to see that, as long as  $[h]_{\sim} \neq \emptyset$ , we have  $[h]_{\sim} = [h']_{\sim}$  if and only if  $[h]_{\dot{\sim}} = [h']_{\dot{\sim}}$ . Inductively it follows that the equivalence of epistemic experiences is invariant under taking the S5 closure as long as  $[h]_{\sim} \neq \emptyset$ ; persistent insanity ensures that  $\text{PR}_{\text{ee}}$  is also satisfied for any  $h'$  extending an  $h$  with  $[h]_{\sim} = \emptyset$ .  $\square$

To see that persistent insanity is indeed needed for this result, consider Figure 4(a) with the accessibilities  $e_2 \sim e_1$  and  $e_1 \sim e_1$  removed. The resulting frame still has  $\text{PR}_{\text{ee}}$ , but its S5 closure does not have  $\text{PR}_{\text{hc}}^{\ell}$ .

<sup>10</sup> Note that this is indeed a local condition: On introspective frames the S5 closure is the symmetric and reflexive closure, without any need of iterating through the accessibility relation (cf. Fact 5.1).

**Corollary 5.4.** *A KD45 ETL frame has  $\text{PR}_{\text{ee}}$  if and only if its S5 closure has  $\text{PR}_{\text{hc}}^\ell$ .*

*Proof.* Immediate, since KD45 frames are introspective and vacuously satisfy persistent insanity.  $\square$

*Remark 5.5.* Note that  $\text{PR}_{\text{hc}}^\ell$  implies persistent insanity, so Proposition 5.3 applies to all introspective frames with  $\text{PR}_{\text{hc}}^\ell$ .

As witnessed by Figure 4 and by Figure 5 later on, the notions  $\text{PR}_{\text{ee}}$  and  $\text{PR}_{\text{hc}}^\ell$  are incomparable in the sense that each one is stronger than the other one under certain circumstances. Proposition 5.3 gives an insight as to why this is so: By applying the  $\text{PR}_{\text{hc}}^\ell$  condition to the S5 closure of a frame, on the one hand the antecedent in this condition becomes more permissive, but on the other hand so does the consequent. Thus, the condition gets both strengthened and weakened.

Now that we have contrasted our basic notions of perfect recall and provided and discussed separate local characterizations, we proceed to characterize the combination of the notions. We use  $\text{PR}$  to denote the combination of perfect recall notions,  $\text{PR}_{\text{ee}}$  plus  $\text{PR}_{\text{hc}}^\ell$  (and thus  $\text{PR}_{\text{hc}}$ ), describing perfect-recall agents that can reason both about their epistemic experience and about history consistency.

As mentioned earlier, our considerations so far hold both for ETL trees and ETL forests. Now, however, the distinction becomes important. We start by focusing on trees.

## 5.4 Characterizing PR on trees

It turns out that on introspective trees,  $\text{PR}_{\text{hc}}^\ell$  captures both definitions of perfect recall, much like it (and  $\text{PR}_{\text{hc}}$ ) did on S5 frames. This allows us to define and axiomatize  $\text{PR}$  on introspective trees, reusing the results we obtained in Section 4.

**Theorem 5.6.** *An introspective ETL tree has  $\text{PR}$  if and only if it has  $\text{PR}_{\text{hc}}^\ell$ .*

Note that this result is not in contradiction with the examples in Figure 4, since (b) is not transitive and (c) is not Euclidean. For the proof, we need the following auxiliary results.

**Observation 5.7.** *For any introspective ETL frame with  $\text{PR}_{\text{hc}}^\ell$  and histories  $h_1, h_2$  with  $h_1 \preceq h_2$  and  $h_1 \sim h_2$ , for each  $h'_1 \preceq h_1$  there is  $h'_2 \preceq h_2$  such that  $h'_1 \sim h'_2$ .*

*Proof.* The claim can be shown with a simple induction on  $h_1$ , using Lemma 4.2 and transitivity of  $\preceq$ .  $\square$

**Lemma 5.8.** *For any introspective ETL frame with  $\text{PR}_{\text{hc}}^\ell$  and histories  $h_1$  and  $h'_2 \preceq h \preceq h_2$ , if  $h_1 \sim h_2$  and  $h_1 \sim h'_2$  then  $h_1 \sim h$ .*

*Proof.* With Euclideaness, we obtain  $h_2 \sim h'_2$ . Let  $h''_2$  be the shortest prefix of  $h_2$  such that  $h_2 \sim h''_2$  (note that  $h''_2 \preceq h'_2 \preceq h$ ). Observation 5.7 implies that there must be  $h' \preceq h''_2$  with  $h \sim h'$ . Another application of Observation 5.7 then yields that there is  $h'' \preceq h'$  such that  $h''_2 \sim h''$ , and by transitivity we get  $h_2 \sim h''$ . Now if  $h' \prec h''_2$  then  $h'' \prec h''_2$ , contradicting that  $h''_2$  is the shortest prefix accessible from  $h_2$ . So  $h' = h''_2$ , thus  $h \sim h''_2$ . Since  $h_1 \sim h_2 \sim h''_2$ , we obtain the claim.  $\square$

**Lemma 5.9.** *If an introspective ETL tree has  $\text{PR}_{\text{hc}}^\ell$ , then so does its S5 closure.*

*Proof.* Take any introspective tree  $\mathcal{F}$  with  $\text{PR}_{\text{hc}}^\ell$ . To see that its S5 closure  $\dot{\mathcal{F}}$  also has  $\text{PR}_{\text{hc}}^\ell$ , let  $h, e, h'$  be such that  $he \sim h'$ . We need to show that  $\sim$  satisfies one of the three conditions in the definition of  $\text{PR}_{\text{hc}}^\ell$ .

Since  $\sim$  is the symmetric and reflexive closure of  $\sim$  (cf. footnote 10), we have either of these cases:

- $he = h'$ . Since  $h \sim h$ , condition (ii) in the definition of  $\text{PR}_{\text{hc}}^\ell$  obtains.
- $he \sim h'$ . Since  $\mathcal{F}$  has  $\text{PR}_{\text{hc}}^\ell$ ,  $\sim$  satisfies one of the three conditions of  $\text{PR}_{\text{hc}}^\ell$ , thus so does  $\sim$ .
- $h' \sim he$ . If  $h' \neq \epsilon$  then the same argument as in the previous case applies. Otherwise,  $h' = \epsilon \sim he$ . Euclideaness of  $\sim$  yields  $he \sim he$ , and since  $\sim$  satisfies  $\text{PR}_{\text{hc}}^\ell$ , we have either of these three cases:
  - (i)  $h \sim he$ . Since  $h' \sim he$ , we get  $h \sim h'$ , so  $\sim$  satisfies condition (i) of  $\text{PR}_{\text{hc}}^\ell$ .
  - (ii)  $h \sim h$ . Since  $h' = \epsilon \preceq h$ , Observation 5.7 yields that there is  $h'' \preceq h$  such that  $h' \sim h''$ . Now we have  $h' \sim he$ ,  $h' \sim h''$  and  $h'' \preceq h \preceq he$ , so Lemma 5.8 applies and yields  $h' \sim h$ . Symmetry of  $\sim$  yields  $h \sim h'$ , so  $\sim$  satisfies condition (i) of  $\text{PR}_{\text{hc}}^\ell$ .
  - (iii)  $he \sim h$ . Together with  $h' \sim he$  we get  $h' \sim h$ . Symmetry of  $\sim$  again yields  $h \sim h'$ , condition (i) of  $\text{PR}_{\text{hc}}^\ell$ .  $\square$

We can now straightforwardly prove the stated result.

*Proof of Theorem 5.6.* PR implies  $\text{PR}_{\text{hc}}^\ell$  by definition. To see that the reverse direction holds, take any introspective ETL tree  $\mathcal{F}$  that has  $\text{PR}_{\text{hc}}^\ell$ . Due to Lemma 5.9, its S5 closure also has  $\text{PR}_{\text{hc}}^\ell$ , and with Proposition 5.3 and Remark 5.5 it follows that  $\mathcal{F}$  also has  $\text{PR}_{\text{ee}}$ .  $\square$

Since the proof of Theorem 4.4 did not use S5, we immediately obtain an axiomatization of ETL on introspective trees with perfect recall. Further, it is easy to see that on synchronous trees, sPR and  $\text{PR}_{\text{hc}}^\ell$  are still equivalent, so we also have an axiomatization of ETL on synchronous introspective trees with perfect recall.



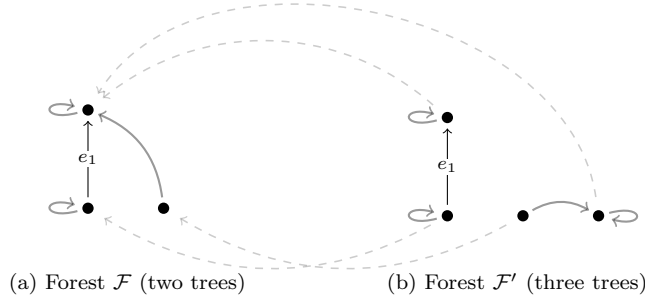


Figure 5: (a)  $\mathcal{F}$  is an introspective forest that has  $\text{PR}_{\text{hc}}^\ell$ , but not  $\text{PR}_{\text{ee}}$ . (b)  $\mathcal{F}'$  has  $\text{PR}_{\text{hc}}^\ell$  and  $\text{PR}_{\text{ee}}$ , and  $\mathcal{F}$  is its bounded morphic image via the bounded morphism depicted with dashed arrows. So PR is not modally definable on forests.

## 5.5 Characterizing PR on forests

Theorem 5.6 does not apply to forests, as witnessed by Figure 5(a). Before we look at how to characterize PR here, we note that, unlike on S5 forests, defining PR on introspective forests generally is impossible in ETL (the same holds for  $\text{PR}_{\text{ee}}$ ).

**Proposition 5.10.** *PR is not modally definable on introspective ETL forests (and thus not on general ETL forests either).*

*Proof.*  $\mathcal{F}'$  in Figure 5 has PR, while its bounded morphic image  $\mathcal{F}$  does not. Since modally definable properties are closed under bounded morphic images (cf. [1]), the claim follows.  $\square$

From Proposition 5.3 and Remark 5.5 it is clear that any introspective frame has PR if and only if both it and its S5 closure have  $\text{PR}_{\text{hc}}^\ell$ . However, with an additional slight restriction, we can continue to use  $\text{PR}_{\text{hc}}^\ell$  to characterize PR. From Figure 5, it is intuitively clear that accessibilities from some root to a later state in some (different) tree are problematic: In such cases,  $\text{PR}_{\text{hc}}^\ell$  is vacuously satisfied, while  $\text{PR}_{\text{ee}}$  may not hold.

To fix this, call an ETL forest  $\mathcal{F}$  **initially synchronous** if, for any two roots  $\epsilon, \epsilon'$  and history  $h$  with  $\epsilon \preceq h$  and  $\epsilon' \sim h$ , we also have  $\epsilon' \sim \epsilon$ . That is, the agent at least considers it possible that indeed no time has passed initially, although he may immediately lose synchronicity and also consider later states possible. We then get the following.

**Lemma 5.11.** *If an introspective and initially synchronous ETL forest has  $\text{PR}_{\text{hc}}^\ell$ , then so does its S5 closure.*

*Proof.* The proof is analogous to that of Lemma 5.9, with one additional observation: If  $\epsilon' \sim h$  for some history  $h$  with root  $\epsilon$ , then initial synchronicity yields  $\epsilon' \sim \epsilon$ . Euclideaness then yields  $\epsilon \sim h$ , and with Theorem 5.6 it follows that  $\text{EE}(\epsilon) \approx \text{EE}(h)$ . Due to Fact 5.1, we also have  $\text{EE}(\epsilon') \approx \text{EE}(\epsilon)$ , so  $\text{EE}(\epsilon') \approx \text{EE}(h)$  by transitivity of  $\approx$ .  $\square$

**Theorem 5.12.** *An introspective and initially synchronous ETL forest has PR if and only if it has  $\text{PR}_{\text{hc}}^\ell$ .*

*Proof.* Analogously to Theorem 5.6, this follows from Lemma 5.11, Proposition 5.3 and Remark 5.5.  $\square$

## 6 Conclusions

We discussed two different ways of “not losing information”, that is, accessing and reasoning with one’s memories. The first one has been the fundamental definition in the literature on perfect recall. It assumes that a perfect-recall agent can use differences in past epistemic states in order to distinguish present states. The second one is a consistency condition on the histories considered possible. While it has been used in the literature as technical condition, we provided it with its own motivation.

The two notions have previously been studied in S5, where they coincide, and in logics with long-range temporal operators. We gave a novel characterization and axiomatization in ETL, using only next-step temporal operators.

We then dropped the assumption of S5 and noticed that the notions no longer coincide. Since they capture two independently motivated ways of reasoning with memories, we examined and characterized them individually as well as jointly.

Given that the two notions use different aspects of ETL models, some discussion is needed concerning the access that we assume an agent to have.

It is a general issue in modeling agents to what extent the model faithfully represents an agent’s internal workings, and to what extent it represents the modeler’s external perspective. What we mean if we say that an agent “does not lose information”, of course, depends on what information we ascribe to him in the first place. ETL is agnostic as to whether the agent has direct access to the semantic structures constituting a model or whether they are just a representation for the modeler, and whether the logic language is supposed to reflect the agent’s “mentalese” or whether it is just a way for the modeler to talk *about* the agent. Depending on the intended interpretation, one may exclude or include certain features in what is considered the agent’s information, and one may accept or reject certain methods for the agent to access and reason with his memories.

Since ETL does not specify these issues, we simply examined what can be said *if* the agent has access to certain aspects of the model. Outside of S5, where the notions do not coincide, it depends on the modeled situation which definition of perfect recall is the right one.

An interesting question for further research is whether there are additional aspects of reasoning with memories, which might be conflated in S5 with the ones we discussed, but distinct in other settings.

The general goal with these considerations is to help improve our understanding of the assumptions implicit in the framework or explicitly made by the

modeler when modeling agents. Along the lines of the inspiring work by van Benthem et al. [18], we hope to obtain more fine-grained insights in more general settings.

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